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The limits of Poisson's ratio in polycrystalline bodies

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While it has been established that the elastic moduli and compliances of anisotropic and isotropic materials should be positive for thermodynamic reasons, no condition related to the values of Poisson's ratio has yet been established. However, it is generally accepted that for isotropic materials Poisson's ratio should vary between − 1.0 and 0.5, whereas for orthotropic materials various conditions have been introduced relating the different components of the anisotropic Poisson's ratio with the remaining elastic constants of the material. In this paper, limits for Poisson's ratio of body-centred cubic (bcc) polycrystalline materials are determined, based on the modes of deformation of a typical unit cell of the material subjected to a uniform external loading arbitrarily oriented relative to the principal axes of the crystal. It is shown that the values of Poisson's ratio thus established correlate satisfactorily with experimental values of this constant. The procedure can be readily applied to other structural units of polycrystalline bodies.

1. Introduction

It has been shown that the spectral decomposition of the compliance fourth-rank tensor, $\boldsymbol{S}$, of a solid, which defines its eigenvalues, expresses necessary and sufficient conditions assuring positive values for the function of the elastic potential. It has been proved that the elastic potential takes positive values, if and only if all the eigenvalues of the tensor $\boldsymbol{S}$ are positive [1–4]. The condition of positiveness of the elastic potential function for the transversely isotropic material implies the positiveness of all eigentensors of the fourth-order compliance tensor of the material $\boldsymbol{S}$, which, in turn, yields the following inequalities [5, 6, 7].

For transversely isotropic bodies it has been shown that the longitudinal ν_L, and the transverse, ν_T, Poisson's ratios should satisfy the following relationships

$$|\nu_T| < 1 \qquad (1)$$

$$\frac{2\nu_L^2}{(1-\nu_T)} < \frac{E_L}{E_T} \qquad (2)$$

where E_L, ν_L and E_T, ν_T are the longitudinal and transverse elastic moduli and Poisson's ratios of the material [8]. Thus, while the positiveness for all the elastic moduli (E_L and E_T) and the respective shear moduli (G_L and G_T) is valid, because of thermodynamic reasons, Poisson's ratios do not satisfy such a simple condition.

For isotropic materials, where it is valid that $\nu_L = \nu_T = \nu$ and $E_L = E_T = E$, Equation 2 yields

$$-1.0 \leq \nu \leq 0.5 \qquad (3)$$

For these materials the well-known relationships between their shear G, and bulk, K, moduli and the elastic modulus, E, and Poisson's ratio, ν, hold [9]

$$G = \frac{E}{2(1+\nu)}, \qquad K = \frac{E}{3(1-2\nu)}$$

Similar expressions for the limits of the values of Poisson's ratios were established previously [5, 6] by equivalent procedures, different than the straightforward analysis based on spectral decomposition of the compliance tensor, $\boldsymbol{S}$, where again the condition that all elastic moduli are positive was used.

However, all materials used in engineering applications present positive values for their Poisson's ratios. Only some foamy materials and some materials of very high porosity present either very low or negative values for Poisson's ratio [10]. But such materials are as yet of very limited interest in engineering applications and therefore we shall dispense with them. Then, Equations 1–3 impose the real limits of variation of Poisson's ratios between zero and 0.5 for isotropic bodies and between zero and unity for the transverse Poisson's ratio of the orthotropic materials; whereas for the respective longitudinal Poisson's ratio, ν_L, Inequality 2 is valid.

TABLE I The values of Poisson's ratio of various substances belonging to different lattice systems

BCC	FCC	HCP	Diamond	Sodium chloride[a]
K, 0.446	Cu, 0.35	Mg, 0.33	Diamond, 0.298	NaCl, 0.207
Na, 0.433	Ni, 0.31	Zn, 0.35	Si, 0.278	BaCl, 0.283
W, 0.283	Al, 0.34	Cd, 0.29	Ge, 0.271	KCl, 0.134
V, 0.360	Pb, 0.45	Co, 0.31		KBr, 0.142
Nb, 0.380	Au, 0.42	Zr, 0.34		KI, 0.137
Fe, 0.373[b]	Ag, 0.38	Ti, 0.34		LiF, 0.312
Cr, 0.382	Pt, 0.39			
	Fe, 0.28[c]			

[a] Sodium chloride ionic solid structures are formed by two intertwined fcc structures for each element.
[b] Low-temperature value.
[c] High-temperature value.

However, Table I extracted from [10] indicates the following facts. (i) The majority of the values for Poisson's ratio lie between $\nu = 0.30$ and $\nu = 0.38$. (ii) There is a rather distinct separation of these values for the unit cells of different space lattices. Thus, the body-centred cubic (bcc) and face-centred cubic (fcc) systems have Poisson's ratios in the same approximate range, whereas the hexagonal close packed (hcp) system elements have rather lower values of Poisson's ratio. Furthermore, the diamond system possesses significantly lower values, whereas sodium-chloride systems, where the crystal is formed by two intertwined fcc systems and is therefore more compactly arranged, have much lower values of Poisson's ratios. These remarks suggest that the values of Poisson's ratios for different materials are highly dependent on the type of structure of the cells.

Finally, it is worthwhile remarking that to ensure the positiveness of the elastic potential function both of Inequalities 1 and 2 should be simultaneously satisfied. Neglecting to satisfy one of these inequalities may lead to erroneous values for Poisson's ratio. As an example, a very high value for Poisson's ratio has been given for orthotropic plates of a boron–epoxy composite, a value of Poisson's ratio of $\nu_L = 1.97$ was given, and this was considered to be acceptable. This was because only one of Inequalities 1 and 2 was satisfied for defining this value of Poisson's ratio [7].

In this paper a study was undertaken to explain the values and the limits of Poisson's ratios of the two more common structures in the metallic crystals, that is the bcc and the fcc cubic systems, as well for hcp systems. Simple analyses of these most common arrangements, along all possible paths of load transmission of their unit cells, explain the phenomenon in which the values of Poisson's ratios are concentrated in more restricted limits than those indicated by the conditions implied by thermodynamic principles.

2. The limits of Poisson's ratio for orthotropic bodies

For a general anisotropic elastic body the requirement implied by thermodynamics is that its elastic potential, Φ, must be positive. This elastic potential is expressed by

$$\Phi = \tfrac{1}{2}\boldsymbol{\sigma}\boldsymbol{S}\boldsymbol{\sigma} = \tfrac{1}{2}\boldsymbol{\varepsilon}\boldsymbol{C}\boldsymbol{\varepsilon} > 0 \qquad (4)$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ express the stress and strain tensors and $\boldsymbol{S}$ and $\boldsymbol{C}$ are the compliance and stiffness symmetric fourth-order tensors, respectively, which should be always positive and definite.

For a transversely isotropic body, which is of interest in applications, the following expressions give the four roots of the minimum polynomial for $\boldsymbol{S}$, which constitute the respective eigenvalues given by [1–4]

$$\lambda_1 = \frac{(1+\nu_T)}{E_T} = \frac{1}{2G_T} \qquad (5a)$$

$$\lambda_2 = \frac{1}{2G_L} \qquad (5b)$$

$$\lambda_3 = \frac{(1-\nu_T)}{2E_T} + \frac{1}{2E_L} + \left\{\left[\frac{(1-\nu_T)}{2E_T} - \frac{1}{2E_L}\right]^2 + \frac{2\nu_L^2}{E_L^2}\right\}^{1/2} \qquad (5c)$$

$$\lambda_4 = \frac{(1-\nu_T)}{2E_T} + \frac{1}{2E_L} - \left\{\left[\frac{(1-\nu_T)}{2E_T} - \frac{1}{2E_L}\right]^2 + \frac{2\nu_L^2}{E_L^2}\right\}^{1/2} \qquad (5d)$$

The indices L and T denote the longitudinal and the transverse directions respectively.

From Equation 5a–d apart from the obvious positiveness of $E_{T(L)}$ and $G_{T(L)}$, the Poisson's ratios have

$$1 > |\nu_T| \qquad (6)$$

$$\frac{E_L}{E_T} > \frac{2\nu_L^2}{(1-\nu_T)} \qquad (7)$$

Equations 6 and 7 were previously established by Eubanks and Sternberg [5] from expressions of the strain-energy density in terms of the components of the stiffness tensor, $\boldsymbol{C}$, and the strain, ε, where the $\boldsymbol{C}$ tensor should satisfy conditions of positiveness.

It is worthwhile indicating that Equation 6 and 7 were established by a straightforward solution of Equation 5 derived by a rational analysis of a spectral decomposition of the compliance tensor, $\boldsymbol{S}$. Furthermore, similar relationships for a general orthotropic material were established by Jones [7] by using a series of conditions relating the diagonal and the off-diagonal components of the $\boldsymbol{S}$ and $\boldsymbol{C}$ tensors and their interrelations with the elastic anisotropic moduli and Poisson's ratios of the body.

These relationships under the condition of positive elastic potential yield the respective relationships.

$$|\nu_{12}| < \left(\frac{E_{11}}{E_{22}}\right)^{1/2} \qquad (8)$$

$$|\nu_{13}| < \left(\frac{E_{11}}{E_{33}}\right)^{2} \qquad (9)$$

$$|\nu_{23}| < \left(\frac{E_{22}}{E_{33}}\right)^{1/2} \qquad (10)$$

and

$$2\nu_{12}\nu_{23}\nu_{13}\frac{E_{33}}{E_{11}} < 1 - \nu_{12}^2\frac{E_{22}}{E_{11}} - \nu_{23}^2\frac{E_{33}}{E_{22}} - \nu_{13}^2\frac{E_{33}}{E_{11}} < 1 \qquad (11)$$

where E_{11}, E_{22} and E_{33} are the elastic moduli for the strongest medium and weakest direction of anisotropy and ν_{ij} $(i, j = 1, 2, 3)$ are the respective values for the anisotropic Poisson's ratios.

It can be readily shown that for the transversely isotropic body, where $E_{11} = E_L$ and $E_{22} = E_{33} = E_T$ and $\nu_{23} = \nu_T$, $\nu_{12} = \nu_{13} = \nu_L$, Inequalities 8–11 turn into Inequalities 6 and 7.

It is worthwhile mentioning that all of Inequalities 8–11 must be satisfied simultaneously in order to assure the positiveness of the elastic potential function of the material.

On the other hand, Christensen [11] developed another group of such relationships by bounding the values of Poisson's ratios through a procedure of maximization of the other elastic constants of the material. The bounding values for the transverse elastic modulus and Poisson's ratio for bounding the longitudinal Poisson's ratio are expressed by

$$E_{\mathrm{T}} = \frac{4G_{\mathrm{T}}K_{\mathrm{T}}}{K_{\mathrm{T}} + G_{\mathrm{T}} + 4\nu_{\mathrm{L}}^2 G_{\mathrm{T}}(K_{\mathrm{T}}/E_{\mathrm{L}})}$$

and (12)

$$\nu_{\mathrm{T}} = \frac{K_{\mathrm{T}} - G_{\mathrm{T}} - 4\nu_{\mathrm{L}}^2 G_{\mathrm{T}}(K_{\mathrm{T}}/E_{\mathrm{L}})}{K_{\mathrm{T}} + G_{\mathrm{T}} + 4\nu_{\mathrm{L}}^2 G_{\mathrm{T}}(K_{\mathrm{T}}/E_{\mathrm{L}})}$$

where K_{T} is the plane-strain bulk modulus of the material and G_{T} is the shear modulus along the transversely isotropic plane of the body. Indeed, Inequality 8 was derived by putting the value $K_{\mathrm{T}} = \infty$ for the plane-strain bulk modulus, K_{T}, and $G_{\mathrm{T}} = \infty$ for the shear modulus.

Then, a rational mathematical formalization was used by Christensen in order to define valid intervals for the components of Poissons ratio. This is a relatively simple procedure and it leads to reasonable results, however, establish rather broad bounds and thus restrict their effective use for defining the correct mechanical behaviour of anisotropic materials.

3. Crystal arrangements for defining closer bounds of Poisson's ratio

The ratio of the transverse strain (contraction or extension) to a longitudinal strain (extension or contraction) which is observed in a tension or compression test was the first attempt to theoretically derive the so-called Poisson's ratio in 1829 by Poisson [12]. This derivation was based on the general equations of equilibrium obtained by a system of particles between which molecular forces of attraction act. For an isotropic body, a value for Poisson's ratio equal to $\nu = 0.25$ was found, and this value was accepted as a universal constant. Since this constant value differed considerably from any value experimentally established, it was accepted that the value of Poisson's ratio is not a universal constant, and must be regarded as a particular property of each material. Nevertheless, extensive experimental evidence has also firmly established that, for the majority of structural materials, Poisson's ratios vary normally between $\nu = 0.33$ and $\nu = 0.40$, thus falling inside a limited bound, which may be established as a zone of almost equal values of this material constant.

Furthermore, measurements of the mechanical properties in single crystals revealed the fact that their properties vary with the direction of loading. Since a polycrystalline body presents a great variety of differently oriented crystals, it is reasonable to accept that the overall macroscopic strain of the isotropic body depends on the statistical contribution of the particular deformations of each and every crystal in the body, thus making the contribution from the orientation of loading and its distribution inside the body a significant factor. Then, an accurate evaluation of Poisson's ratio values necessitates consideration of this influence. We shall subsequently consider the deformation distribution inside a crystalline unit corresponding to one of the commonly found crystalline arrangements, that is the bcc and fcc systems as well as the hcp system.

Furthermore, atomic arrangements can be found in a number of crystalline bodies, where the atoms are packed as tightly as possible, occupying 74% of the available space. Each atom in these arrangements is a part of a layer of atoms and two successive layers are combined, so that the atoms of one layer rest on the depressions between atoms of the neighbouring layer. A third successive layer can be added, giving either the fcc or bcc structure.

The first layer of atoms is visualized as being on a flat surface (layer A), each atom touching its nearest neighbours. The next layer of atoms (layer B) is placed with each atom nestled into the depressions between the atoms of layer A. The third successive layer also occupies the depressions of the atoms of layer B. However, there are two available positions, either directly over the atoms of layer A, or displaced diagonally to the next empty depressions of layer B (layer C). Thus, the sequence ABABAB... of layers produces the hcp system, whereas the sequence ABCABC... produces the fcc structure. Both structures correspond to the densest possible packing of spheres. Each atom in both packing systems is surrounded by 12 identical atoms; six in the same layer, three above and three below. Spaces among the atoms are surrounded by either four atoms (tetrahedral voids) or by six atoms (octahedral voids).

Thus, the derivation of Poisson's ratio will be undertaken under the following assumptions

1. A linear stress–strain relationship will be postulated between the atoms having closest spacings.
2. The two most important practical crystal structures will be studied. These are the hcp and the fcc systems.
3. The extreme directions of loading relative to the orientation of the crystal-lattice axes will be considered.
4. A single structural element will be isolated and analysed as a space framework to derive deflections from the externally applied loads.
5. Poisson's ratio will be computed for a single element, whereas the effect of the orientation of loading will be evaluated statistically for the whole of a polycrystalline body.
6. The actual crystal will be assumed to be composed of many elements.
7. The applications of the results for single crystals will be incorporated into a general theory of polycrystalline bodies.

The bcc lattice, applying to only a few metals of practical interest, will not be considered in detail in this study, although the particular analyses encountered for the two other systems may be used for evaluating Poisson's ratio for such materials.

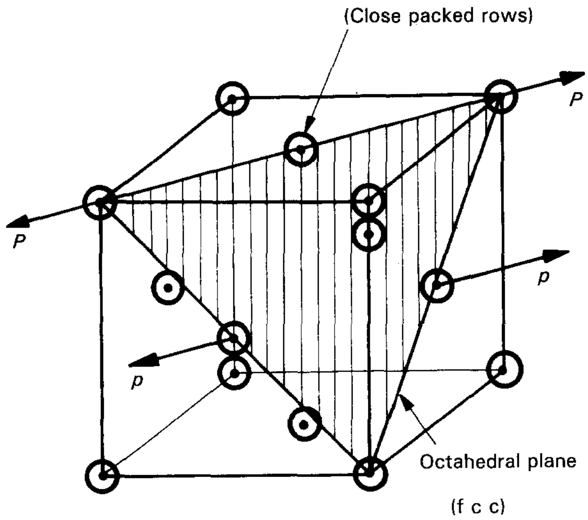


Figure 1 The fcc crystal lattice.

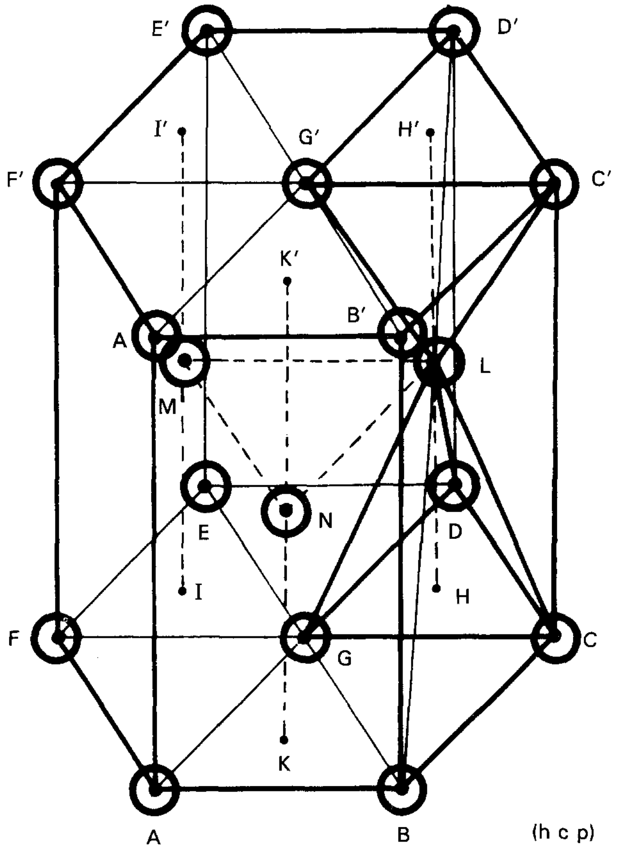


Figure 2 The hcp crystal lattice.

The fcc and hcp lattices are systems with the closest possible packing of atoms, and since they have the same density they allow use of structural elements composed of members having the same length, representing the equilibrium distance between atoms when the system is unloaded. Figs 1 and 2 represent the arrangements of the rows of atoms in the crystal lattice for the fcc the hcp systems, respectively.

For a better understanding of the particular characteristics of both structures and to give a convenient picture for analytical purposes, we construct a three-dimensional model composed of unit spheres representing the atoms at the corners of the lattice. We shall then proceed to examine an actual pyramid of spheres in the two systems, by loading in a direction normal to the planes of the arrays of spheres and applying a vertical force to these pyramidal piles of spheres, which can be transmitted through the constraints of the framework. Thus, Fig. 3 represents a typical projection for the fcc system on the diagonal (ADGFA)-plane of the unit pyramid (AILMKNJG) containing two opposite tetrahedra AILM and GJKL, connected by two square pyramids KLMJN and ILMJN. The external force $P = 1$ is applied along the diagonal AG of the unit cube.

Projections of all the unit-spheres lying outside the diagonal plane ADGF on this plane yield the plane frame shown in Fig. 4b; Fig. 4a shows the projection of this frame unit, on any plane, of the layers of close-packed atoms.

While Fig. 4 corresponds exclusively to the fcc system, Fig. 5 shows the respective frame unit for the hcp system, which is much simpler than the corresponding frame unit for the fcc system in Fig. 4. Again, Fig. 5a1 shows the projection of this frame on the plane of close-packed spheres, whereas Fig. 5a2 shows the projection along the respective diagonal plane normal to the previous one.

Further examination of both piles for the bcc and the hcp crystal lattices indicates that in either system there are complementary structural elements consisting of two square pyramids joined at their bases, as indicated in Fig. 5b. Fig. 5b1 shows the projections on both pyramids on their common base, whereas Fig. 5b2 shows the projection on the diagonal plane.

Furthermore, it can also be seen that in both crystal lattices there are straight rows of atoms in several planes, which are capable of transmitting axial forces (see Fig. 1). The relationship of these models to the unit cells, shown in Figs 1 and 2 for the fcc lattice and for the hcp lattice shown in Figs 3 and 2, respectively, are indicated by thick lines inside the unit cubic or hexagonal system.

We examine now the models established in Figs 4 and 5 as isolated independent structures subject to axial tensile loads of unit intensity.

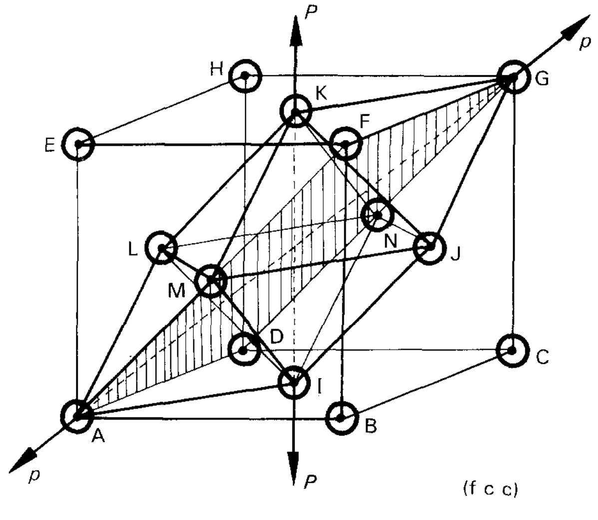


Figure 3 The existing model units inside the fcc crystal lattice.

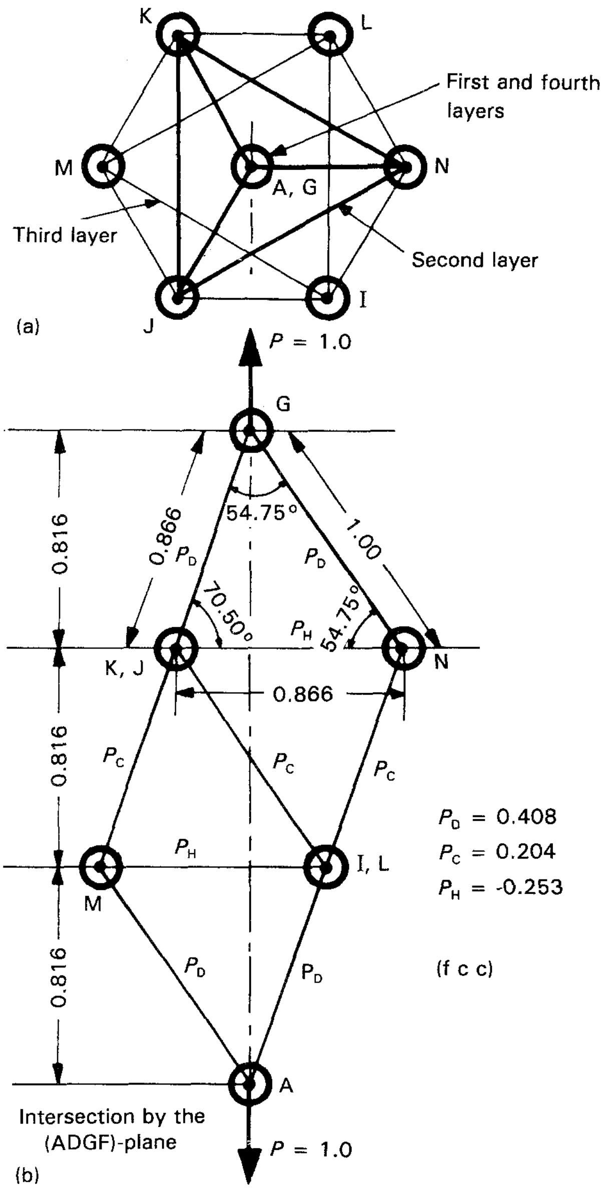


Figure 4 Diagonal and transverse projections of the fcc crystal lattice viewed along the diagonal plane (ADGF) of Fig. 3 for the fcc configuration.

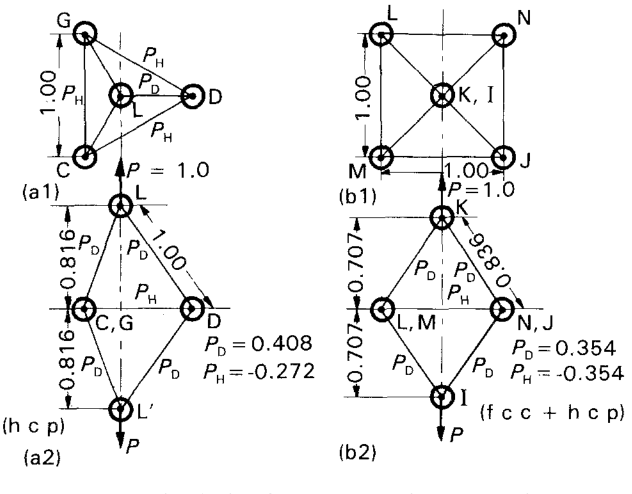


Figure 5 (a) Longitudinal and transverse projections of a hcp crystal lattice, and (b) the same projections for complementary model units in both the fcc and the hcp crystal lattices.

4. Evaluation of Poisson's ratios of typical isolated structures

We study now the deflections of each particular structure corresponding to either lattice. The axial and transverse deflections of each beam of the elementary framework will be calculated by the classical methods of analysis. These deflections will be divided by the lengths of each beam to determine the strain. Poisson's ratio will be derived by the ratio of the overall transverse strain to the respective axial strain for each model unit, assuming that the deflection of each member is proportional to the transmitted force.

From the equations of equilibrium it is found that the forces in the diagonal and horizontal members of the elementary unit of Fig. 4 are given by

$$P_D = 0.408\,P,\quad P_C = 0.204\,P,\quad P_H = -0.253\,P \qquad (13)$$

In this framework all the P_D, P_C and P_H members are six in number for each unit.

Since all members have the same length, Hooke's law is expressed by $\delta_i = P_i/E'$ (with i = D, C, H), which δ is the deflection of the member, P is the force in the member and E' is the constant elastic modulus of each member, multiplied by its cross-section, which is also taken as being the same for all members.

The contribution of each of the above deflections to the overall vertical deflection is found by applying Castigliano's theorem. The coefficients thus obtained are numerically equal to the forces in the members by assuming that the applied external force P is equal to unity.

Then, the vertical deflection is given by

$$\delta_v = [6\times 0.408^2 + 6\times 0.204^2 + 6\times(-0.253)^2]\frac{P}{E'}$$

$$\delta_v = 1.633\frac{P}{E'} \qquad (14)$$

Then, the axial strain is given by

$$\varepsilon_v = \frac{\delta_v}{L_v} = \frac{1.633}{2.448}\frac{P}{E'} = 0.667\frac{P}{E'} \qquad (15)$$

where L_v is the vertical length of the element in Fig. 4, which is equal to $L_v = 2.448$.

Similarly, the horizontal deflection is equal to the respective strain, because the length of the member is equal to unity. Then, it is valid that

$$\varepsilon_H = -0.253\frac{P}{E'} \qquad (16)$$

Finally, the Poisson's ratio of the element of Fig. 4 is given by

$$\nu = -\frac{\varepsilon_H}{\varepsilon_v} = 0.380 \qquad (17)$$

A similar analysis of the element of Fig. 5a yields

$$P_D = 0.408P,\qquad P_H = -0.272\,P \qquad (18)$$

The deflections of the various elements of this model are

$$\delta_D = 0.408\frac{P}{E'},\qquad \delta_H = -0.272\frac{P}{E'} \qquad (19)$$

Thus, the vertical deflection of the model consisting of six members with δ_D and another three members with δ_H is given by

$$\delta_v = [6\times(0.408)^2 + 3\times(-0.272)^2]\frac{P}{E'} \quad (20)$$

Because the vertical length L_v of the overall element is now equal to $L_v = 1.632$ (see Fig. 5a2) the vertical component of strain, ε_v, is given by

$$\varepsilon_v = \frac{\delta_v}{L_v} = \frac{1.223}{1.632} = \frac{P}{E'} = 0.750\frac{P}{E} \quad (21)$$

whereas the horizontal component of strain, ε_H, is given by ($L_H = 1.0$)

$$\varepsilon_H = -0.272\frac{P}{E'} \quad (22)$$

Finally Poisson's ratio of the model of Fig. 5a is given by

$$\nu = -\frac{\varepsilon_H}{\varepsilon_v} = \frac{0.272}{0.750} = 0.353 \quad (23)$$

It is worthwhile noticing that the horizontal strain, ε_H, given in Equation 22 is the same for each element of the triangle representing the cross-section in Fig. 5a1 and therefore Mohr's circle for strain must be a point, showing that the strain is the same in any direction through the cross-section.

Analysis of the element of Fig. 5b which is common to both lattices gives the following results

$$P_D = 0.354P, \quad P_H = -0.354P \quad (24)$$

Moreover, the P_D members are eight in number, whereas the P_H members are four in number then we have

$$\varepsilon_v = 1.06\frac{P}{E'}, \quad \varepsilon_H = -0.354\frac{P}{E'} \quad (25)$$

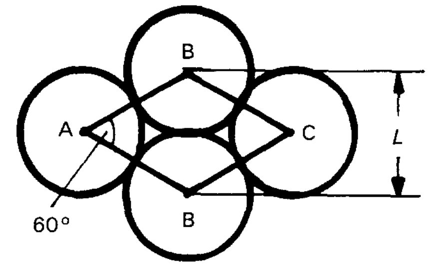
and, finally, a value for Poisson's ratio is given by

$$\nu = -0.355 \quad (26)$$

Finally, there are, in both systems, straight rows of atoms capable of transmitting tensile forces. In the fcc system each row represents the intersection of three planes of atoms with an identical and staggered arrangement, so that they may be thought of as having a three-dimensional symmetry with respect to neighbouring atoms. It is therefore possible to analyse it in a member of a plane structure consisting of a rhomboidal arrangement of atoms, as shown in Fig. 6a, isolated by selecting two atoms together with their closest neighbours in the plane. When the structure is loaded the diagonal members must remain unloaded with their intercentre distances unchanged, because no deformation is accepted for the atoms. However, the axial direction of the unit is lengthened because of the applied axial load, as in Fig. 6b.

By assuming a certain deflection in the axial member the lateral deflection can be readily found. A lateral tension force $P = 1$ will produce a force of $-0.577P$ in the axial member and therefore the lateral deflection δ_H is given by

$$\delta_H = -0.577\delta_v \quad (27)$$



Unit element of single rows of atoms

(a)

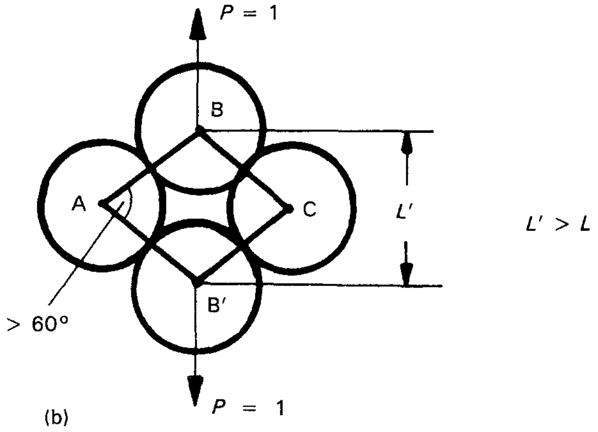


Figure 6 The octahedral plane arrangement of the close-packed rows for both lattices: (a) unloaded, (b) loaded.

Therefore, the axial and lateral components of strain can be evaluated, because the respective lengths are $L_v = 1$ and $L_H = 1.732$ and they are given by

$$\varepsilon_v = \frac{\delta_v}{L_v} = \delta_v$$

$$\varepsilon_H = \frac{-0.577\delta_v}{L_H} = \frac{-0.577\delta_v}{1.732}$$

$$= -0.333\delta_v \quad (28)$$

The respective Poisson's ratio is, therefore, equal to

$$\nu = -\frac{\varepsilon_H}{\varepsilon_v} = 0.333 \quad (29)$$

It follows that because ν is the ratio of two elastic strains it is independent of any law of elasticity which is valid for both strains.

It has been established by this simple framework analysis that the various structural models, which can be isolated in either of the crystal systems studied, have values of Poisson's ratios varying between 0.333 and 0.380. These values represent the extreme values defining the range of variation of Poisson's ratio, as the orientation of the crystal inside the polycrystalline body changes randomly.

When the tension force is applied in an arbitrary direction in the force-transmission path along some axis lying between the extreme axes investigated in the previous analysis it is expected that the respective values for Poisson's ratios will lie inside the established interval, because, if the force assumed is resolved into components along the closest model axes, the cosines of the angles involved would not differ significantly from unity.

Because for the evaluation of Poisson's ratios we are not concerned with absolute values, but with ratios, it appears that the foregoing analysis and results will not appreciably change when all possible directions of loading of the crystal are considered.

The actual crystal may be thought of as a collection of structural models in the lattice, having both regularity and continuity, therefore all the above analysis is valid for the actual crystals of the body. Therefore, whenever forces are transmitted through a crystal along diagonal paths, the Poisson effect is actually caused by surface forces, which, in the case of tension represent compressive stresses which are approximately equal to one-third of the applied tension stress; an exception is that when the force is transmitted along straight rows of atoms then these are not diagonal forces. The lateral contraction occurs because there is no external surface force acting upon the specimen. Hence there is no force at the diagonals. The interatomic bond is still the fundamental cause of the Poisson effect, because, if it were not present along the diagonal axes, there would be nothing to prevent elongation of these axes.

The use of the individual models for the analysis of loading of a polycrystalline body can now be seen to be a convenient artifice replacing the truly typical behaviour of the entire crystal. Actually, the forces could be traced along diagonal paths through the three-dimensional crystal until they arrive at a surface, where they would have to find a new path along different diagonals. The transverse force required for equilibrium at the surface would then be applied across the entire crystal in order to obtain the transverse deflection.

The fact that transverse forces can actually exist when the external forces are purely axial has a practical significance because for a material with low tensile strength in the transverse direction there is a danger of failure by premature longitudinal cracking.

Furthermore, it is worthwhile indicating that the interior state of stress in simple tension is not a simple tension, because if the crystal is oriented in such a way that the tensile forces must be transmitted by diagonal action, it is possible that, at any interior point, the state of stress is axial tension combined with radial compression. Therefore, if the value of Poisson's ratio is assumed to be equal to one-third for computing the transverse effects, all shear stresses would be increased by the same amount.

5. The effect of orientation

Consider a body consisting of a large number of crystals randomly oriented. The crystals are identical, with modulus E' and Poisson's ratio ν'. Consider also a field of uniaxial tension in the x-direction and a crystal with axes x_θ, y_θ, positioned at an angle θ in relation to the axes x, and y. This crystal is subjected to: (i) an axial, stress $\sigma_{x\theta} = \sigma \cos\theta$; and (ii) a transverse stress $\sigma_{y\theta} = \sigma \sin\theta$.

Owing to the stress $\sigma_{x\theta}$ we get (i) an axial strain

$$\Delta_{x\theta\,(1)} = \frac{\sigma}{E'}\cos\theta$$

and (ii) a transverse strain

$$\Delta_{y\theta\,(1)} = -\frac{\sigma}{E'}\sin\theta$$

Owing to the stress $\sigma_{y\theta}$ we get (i) a transverse strain

$$\Delta_{y\theta\,(2)} = \frac{\sigma}{E'}\sin\theta$$

and (ii) an axial strain

$$\Delta_{x\theta\,(2)} = -\frac{\nu'\sigma}{E'}\sin\theta$$

The total strains in the x_θ and y_θ directions are given respectively by

$$\Delta_{x\theta} = \Delta_{x\theta\,(1)} + \Delta_{x\theta\,(2)} = \frac{\sigma}{E'}(\cos\theta - \nu'\sin\theta)$$

and (30)

$$\Delta_{y\theta} = \Delta_{y\theta\,(1)} + \Delta_{y\theta\,(2)} = \frac{\sigma}{E'}(\sin\theta - \nu'\cos\theta)$$

Referring to the x- and y-axis, we get for the strains

$$\Delta_{xn} = \frac{\sigma_0}{E'}(\cos\theta - \nu'\sin\theta)\cos\theta$$

and (31)

$$\Delta_{yn} = \frac{\sigma_0}{E'}(\sin\theta - \nu'\cos\theta)\sin\theta$$

Owing to the large number of crystals and their random orientation we can assume a uniform angular distribution over the entire interval $(0, \pi/2)$.

The overall strains will be

$$\Delta x = \frac{2}{\pi}\int_0^{\pi/2} \frac{\sigma}{E'}(\cos\theta - \nu'\sin\theta)\cos\theta\,d\theta$$

$$\Delta y = \frac{2}{\pi}\int_0^{\pi/2} \frac{\sigma}{E'}(\sin\theta - \nu'\cos\theta)\sin\theta\,d\theta \qquad (32)$$

If, on the other hand, a homogeneous and isotropic material, with respective parameters E and ν, is considered, subjected to a uniaxial tension, σ, we get

$$\Delta x = \frac{\sigma}{E}$$

$$\Delta y = -\nu\frac{\sigma}{E} \qquad (33)$$

and

$$\nu = -\frac{\Delta y}{\Delta x} \qquad (34)$$

The overall value of Poisson's ratio, ν, is then expressed by

$$\nu = -\frac{\int_0^{\pi/2} (\sin\theta - \nu' \cos\theta)\sin\theta \, d\theta}{\int_0^{\pi/2} (\cos\theta - \nu' \sin\theta)\cos\theta \, d\theta} \tag{35}$$

which leads to the expression

$$\nu = \frac{(2 - \pi/2)}{(\pi/2 - \nu')} = \frac{0.429}{1.571 - \nu'} \tag{36}$$

Equation 36 is shown in Fig. 7. It can be seen that if the Poisson ratio, ν', for the individual crystal varies in the range (0.333, 0.380), as has been already established there is hardly any change in the Poisson's ratio of the polycrystalline body, ν, since the $\nu = f(\nu')$ curve in a range from 0.347–0.360 is much smaller than the range (0.333, 0.380) for the individual units.

All practical structures are made from polycrystalline materials, where the crystals are found in a more or less random orientation. If the orientation is truly random it has been previously shown that Poisson's ratio is governed by the integrated action of all the crystals. This value does not greatly vary and may fluctuate over a very limited spectrum of values. It is therefore reasonable to conclude that a single-crystal value expresses, to a good approximation, the respective value for the polycrystalline body, and these values do not diverge significantly from one material to the other; however, a unique value is satisfactory for all materials with the same crystal lattice. Furthermore, there is a small difference between the fcc and the hcp crystals, which represent the majority of the structural metals.

Table I indicates that for any of the actually important crystal-lattice systems the differences between the mean values of Poisson's ratio for each system, although distinct, do not differ significantly

Even when there is a certain degree of preferred orientation, there seems to be no reason to believe that the value of Poisson's ratio might appreciably be affected, even though this might have a considerable effect on the other elastic constants.

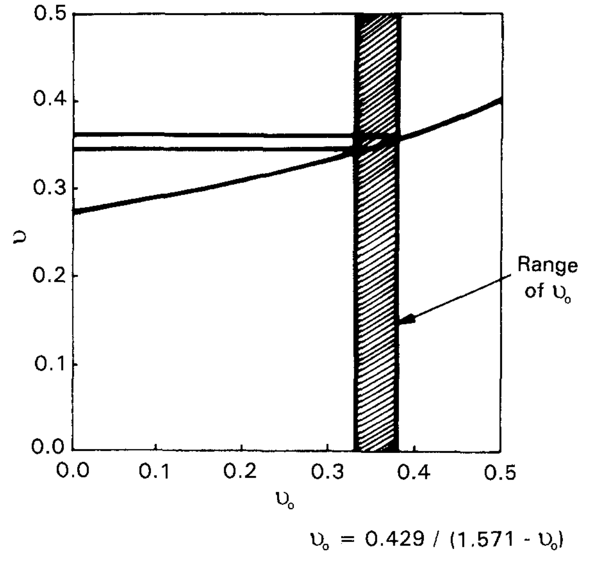


Figure 7 The variation of Poisson's ratio, ν, of a polycrystalline body in terms of the respective Poisson's ratio ν' for the models contained in each system.

Conclusions

1. The value of Poisson's ratio has been calculated for models representing various paths of force transmission through crystals. A linear force–deflection relationship was assumed. The calculated values for fcc and hcp crystals were of the order of one-third, which is close to actual measured values for polycrystalline materials.

2. The value of Poisson's ratio, calculated in the above manner, does not appear to vary greatly with the orientation of the loading.

3. Whenever the force is transmitted through a crystal by diagonal paths, there will be a surface stress acting so as to cause transverse strains. Some observed phenomena (such as the longitudinal cracking of brittle compression specimens, or the longitudinal wrinkling of thin tension membranes) may be explainable on this basis.

4. Similar studies should be made for other crystal lattices and for common alloys.

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References

1. P. S. THEOCARIS, *Proc. Nat. Acad Athens* **64** (1989) 80–100.
2. P. S. THEOCARIS and Th. PHILIPPIDES, *Acta Mechanica* **85** (1990) 13–26.
3. P. S. THEOCARIS, "Constantin Caratheodory: an international tribute", edited by Th. Rassias, Vol. II, (World Scientific, Singapore, 1991) 1354–1377.
4. P. S. THEOCARIS and Th. PHILIPPIDES, *Zeitsch. Angew-Math. Mech.* **71** (1991) 161–171.
5. R. A. EUBANKS and E. STERNBERG, *J Ration. Mech. Analysis* **3** (1954) 89–101.
6. B. M. LEMPRIERE, *AIAA J.* **6** (1968) 2226–2227
7. R. M. JONES, "Mechanics of composite materials" (McGraw-Hill, Kogakusha, Tokyo, 1975).
8. *Idem., J Strain Anal.* **27** (1992) 43–44.
9. A. H. COTTRELL, "The mechanical properties of matter" (Wiley, New York 1964).
10. R. F. ALMGREN, *J. Elasticity* **15** (1985) 427–430.
11. R. M. CHRISTENSEN, "Mechanics of composite materials" (J. Wiley, New York, 1979).
12. S. TIMOSHENKO, "History of strength of materials" (McGraw-Hill, New York 1953).

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